International Journal of Theoretical Physics, Vol. 38, No. 9, 1999

Dirac Equation in an Axial Coulomb Potential

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Received January 5, 1999

We consider solutions to the Dirac equation in the presence of an external axial vector potential $A_\mu(x) = (0, e/|\bar{r}|)$ coupled to the spinor field ψ through the interaction term $-e\bar{\psi}\gamma_\mu\gamma_5\psi A^\mu$. There turn out to be no bound-state energies in this system consistent with a normalizable wave function.

1. INTRODUCTION

It is well known that the axial vector coupling of a quantized spinor field is peculiar, as the divergence of the axial current develops anomalous terms not anticipated when one naively considers the field equations [1–4]. In this paper we consider a somewhat simpler problem. The spinor field is not quantized: we simply examine the solution of the Dirac equation in the presence of a background Coulomb field $A_\mu(x) = (0, e/|\bar{r}|)$ when the coupling of the axial vector field $A_\mu(x)$ to the spinor field $\psi(x)$ is through an axial vector interaction $\mathcal{L}_I = -e\bar{\psi}\gamma_\mu\gamma_5\psi A^\mu$. This problem is very similar to the one addressed by Darwin [5] and Gordon [6]; that is, solving the Dirac equation in an external Coulomb potential in quantum electrodynamics where the coupling is purely vector. However, the absence of parity conservation due to the presence of γ_5 in the interaction $\mathcal{L}_I$ means that, unlike quantum electrodynamics, eigenstates of the Hamiltonian operator need not be parity eigenstates. This we show has the consequence that for no value of the energy eigenvalues can the norm of the wave function be bounded. In quantum electrodynamics, on the other hand, the requirement that the norm of the wave function be bounded results in the usual bound-state energy eigenvalues. The notation of ref. 7 is used.

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0020-7748/99/0900-2371$16.00/0

2. THE AXIAL COULOMB POTENTIAL

The Hamiltonian we consider is

$$\left[\bar{\alpha}\cdot\bar{p}+\beta m-\frac{\alpha}{r}\gamma_5\right]\psi(\bar{r})=E\psi(\bar{r}) \tag{1}$$

where $\bar{p}=i\nabla$, $\alpha=e^2$, and

$$\bar{\alpha}=\begin{pmatrix}0 & \bar{\sigma}\\ \bar{\sigma} & 0\end{pmatrix},\qquad \beta=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\qquad \gamma_5=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \tag{2}$$

Following the discussion in ref. 7, we expand the Dirac spinor field in terms of angular momentum eigenstates so that

$$r\psi_{jm}(\bar{r})=\begin{pmatrix}iG_j^+(r)\phi_{jm}^+(\theta,\phi)+iG_j^-(r)\phi_{jm}^-(\theta,\phi)\\ F_j^+(r)\phi_{jm}^-(\theta,\phi)+F_j^-(r)\phi_{jm}^+(\theta,\phi)\end{pmatrix} \tag{3}$$

The properties of the basis functions $\phi_{jm}^{\pm}$ in (3) reduce our dynamical equation (1) to

$$(m-E)G_j^+ + (-F_j^{+\prime}-(j+\tfrac{1}{2})F_j^+/r)+i\alpha F_j^-/r=0 \tag{4a}$$

$$(m-E)G_j^- + (-F_j^{-\prime}+(j+\tfrac{1}{2})F_j^-/r)+i\alpha F_j^+/r=0 \tag{4b}$$

$$(m+E)F_j^- + (-G_j^{-\prime}-(j+\tfrac{1}{2})G_j^-/r)+i\alpha G_j^+/r=0 \tag{4c}$$

$$(m+E)F_j^+ + (-G_j^{+\prime}+(j+\tfrac{1}{2})G_j^+/r)+i\alpha G_j^-/r=0 \tag{4d}$$

The absence of parity conservation in (1) results in a coupling between the functions (F_j^+, G_j^+) and (F_j^-, G_j^-) when $\alpha\neq 0$.

By examining (4) in the limit $r\to\infty$ we find that

$$F_j^{\pm}(r)=e^{-\sqrt{m^2-E^2}\,r}f_j^{\pm}(r) \tag{5a}$$

$$G_j^{\pm}(r)=e^{-\sqrt{m^2-E^2}\,r}g_j^{\pm}(r) \tag{5b}$$

where $f_j^{\pm}(r)$ and $g_j^{\pm}(r)$ are polynomials in r. Upon making the expansions

$$f_j^{\pm}(r)=\sum_{p=0}^{\infty}\phi_p^{\pm j}r^{s+p} \tag{6a}$$

$$g_j^{\pm}(r)=\sum_{p=0}^{\infty}\gamma_p^{\pm j}r^{s+p} \tag{6b}$$

we obtain the recursion relations from (4),

$$[\pi_p+\kappa]\phi_p^{+j}-i\alpha\phi_p^{-j}-(m-E)\gamma_{p-1}^{+j}-\sqrt{m^2-E^2}\,\phi_{p-1}^{+j}=0 \tag{7a}$$

$$[\pi_p-\kappa]\phi_p^{-j}-i\alpha\phi_p^{+j}-(m-E)\gamma_{p-1}^{-j}-\sqrt{m^2-E^2}\,\phi_{p-1}^{-j}=0 \tag{7b}$$

$$[\pi_p + \kappa]\gamma_p^{-j} - i\alpha\gamma_p^{+j} - (m + E)\phi_{p-1}^{-j} - \sqrt{m^2 - E^2}\,\gamma_{p-1}^{-j} = 0 \quad (7c)$$

$$[\pi_p - \kappa]\gamma_p^{+j} - i\alpha\gamma_p^{-j} - (m + E)\phi_{p-1}^{+j} - \sqrt{m^2 - E^2}\,\gamma_{p-1}^{+j} = 0 \quad (7d)$$

where $\pi_p = p + s$ and $\kappa = +(j + 1/2)$. Since $\phi_{-1}^{\pm j} = 0 = \gamma_{-1}^{\pm j}$, the term in (7) with $p = 0$ is consistent only if

$$s = \pm\sqrt{\kappa^2 - \alpha^2} \quad (8)$$

The negative root is excluded to maintain suitable behavior in (6) at $r = 0$.

Together, (7a) and (7d) imply that

$$[\pi_p + \kappa]\phi_p^{+j} - i\alpha\phi_p^{-j} - \sqrt{\frac{m - E}{m + E}}\{[\pi_p - \kappa]\,\gamma_p^{+j} - i\alpha\gamma_p^{-j}\} = 0 \quad (9a)$$

and similarly from (7b) and (7c),

$$[\pi_p - \kappa]\,\phi_p^{-j} - i\alpha\phi_p^{+j} - \sqrt{\frac{m - E}{m + E}}\{(\pi_p + \kappa)\gamma_p^{-j} - i\alpha\gamma_p^{+j}\} = 0 \quad (9b)$$

Together (9a) and (9b) can be solved to yield

$$\phi_p^{\pm j} = \frac{1}{\pi_p^2 - \kappa^2 + \alpha^2}\sqrt{\frac{m - E}{m + E}}\{[(\pi_p \mp \kappa)^2 + \alpha^2]\,\gamma_p^{\pm j} \pm 2i\alpha\kappa\gamma_p^{\mp j}\} \quad (10)$$

Using (10) and the analogous equation for $\phi_{p-1}^{\pm j}$, we can express 7(c) and 7(d) entirely in terms of $\gamma_p^{\pm j}$ and $\gamma_{p-1}^{\pm j}$,

$$(\pi_p \mp \kappa)\gamma_p^{\pm j} - i\alpha\gamma_p^{\mp j} - \sqrt{m^2 - E^2}\left\{\left[1 + \frac{(\pi_{p-1} \mp \kappa)^2 + \alpha^2}{\pi_{p-1}^2 - \kappa^2 + \alpha^2}\right]\gamma_{p-1}^{\pm j} \pm \frac{2i\alpha\kappa}{\pi_{p-1}^2 - \kappa^2 + \alpha^2}\,\gamma_{p-1}^{\mp j}\right\} = 0 \quad (11)$$

Solving the two equations contained in (11) shows that

$$\gamma_p^{\pm j} = \frac{2\sqrt{m^2 - E^2}}{(\pi_p^2 - \kappa^2 + \alpha^2)(\pi_{p-1}^2 - \kappa^2 + \alpha^2)} \times \{[(\pi_p \pm \kappa)(\pi_{p-1}^2 \mp \pi_{p-1}\kappa + \alpha^2) \mp \kappa\alpha^2]\gamma_{p-1}^{\pm j} + i\alpha[(\pi_{p-1}^2 \pm \pi_{p-1}\kappa + \alpha^2) \pm \kappa(\pi_p \pm \kappa)]\gamma_{p-1}^{\mp j}\} \quad (12)$$

It is evident that no value of E can be selected which ensures that the coefficient of both γ_{p-1}^{+j} and γ_{p-1}^{-j} simultaneously vanish in (12). Consequently, the series in (6) will not terminate, and for large p

$$\gamma_p^{\pm j} \sim \frac{2\sqrt{m^2 - E^2}}{p}\,\gamma_{p-1}^{\pm j} \tag{13}$$

implying that

$$g_j^{\pm}(r) \sim \exp\,[2\sqrt{m^2 - E^2}\,r] \tag{14}$$

Hence the functions $G_j^{\pm}(r)$ in (5b) will grow exponentially as r becomes large and thus the wave function ψ in (1) is unbounded.

The axial vector coupling in Eq. (1) can be supplemented by a vector coupling: *viz* we can replace $-\alpha\gamma_5/r$ by $-\alpha\gamma_5/r - \beta/r$. The procedure outlined above can be repeated without difficulty. If we again use the expansions of Eqs. (3), (5), and (6), then Eqs. (7a)–(7d) are supplemented by the terms $\beta\gamma_p^{+j}$, $\beta\gamma_p^{-j}$, $-\beta\phi_p^{-j}$, and $-\beta\phi_p^{+j}$, respectively. This results in (11) being modified to become

$$\begin{aligned}
&(\pi_p \mp \kappa)\gamma_p^{\pm j} - i\alpha\gamma_p^{\mp j} - \frac{\beta}{\Delta_p}\Bigg[\left(\sqrt{\frac{m-E}{m+E}}\,((\pi_p \mp \kappa)^2 + \alpha^2 - \beta^2)\right.\\
&\quad \left. - \frac{-2E\beta(\pi_p \mp \kappa)}{m+E}\right)\gamma_p^{\pm j} - 2i\alpha\left(\mp\kappa\sqrt{\frac{m-E}{m+E}} + \frac{\beta m}{m+E}\right)\gamma_p^{\mp j}\Bigg]\\
&\quad - \frac{m+E}{\Delta_{p-1}}\Bigg[\left(\sqrt{\frac{m-E}{m+E}}((\pi_{p-1} \mp \kappa)^2 + \alpha^2 - \beta^2) - \frac{2E\beta(\pi_{p-1} \mp \kappa)}{m-E}\right)\gamma_{p-1}^{\pm j}\\
&\quad - 2i\alpha\left(\mp\kappa\sqrt{\frac{m-E}{m+E}} + \frac{\beta m}{m+E}\right)\gamma_{p-1}^{\mp j}\Bigg] - \sqrt{m^2 - E^2}\,\gamma_{p-1}^{\pm j} = 0
\end{aligned} \tag{15}$$

where

$$\Delta_p = \left(\pi_p + \beta\sqrt{\frac{m-E}{m+E}}\right)^2 + \alpha^2 - \kappa^2$$

The two equations contained in (15) are of the form

$$\mathbf{A}_p \bar{\bar{x}}_p = \mathbf{B}_p \bar{\bar{x}}_{p-1} \tag{16}$$

where

$$\bar{\bar{x}}_p = \begin{pmatrix} \gamma_p^{+j} \\ \gamma_p^{-j} \end{pmatrix}$$

In order for $\bar{\bar{x}}_p$ to vanish for some p, the matrix $\mathbf{B}_p$ must vanish as well. In principle, this fixes the value of E, but it is easily seen from (15) that such a value of E does not exist so long as α is nonzero. Hence, irrespective of the strength of the vector coupling, there do not exist normalizable bound-state solutions to the Dirac equation when the external potential is generated by a Coulomb source, provided there is an axial coupling.

3. DISCUSSION

We have demonstrated that the coupling to a massless axial vector in the relativistic quantum mechanical theory of spin-1/2 particles does not lead to a normalizable bound-state wave function when there is an external Coulomb potential. The phenomenological significance of this result is not clear in light of the existence of the Z boson.

We have considered the axial vector field A_μ to which the spinor ψ is coupled to be a fixed classical background field, so strictly speaking we need not concern ourselves with axial gauge invariance. However, if $A_\mu(x)$ is taken to be an interacting field, then it would appear that the absence of axial gauge invariance

$$A_\mu \to A_\mu - \partial_\mu \theta \tag{17a}$$

$$\psi \to e^{i\theta\gamma_5}\psi \tag{17b}$$

$$\overline{\psi} \to \overline{\psi}\, e^{i\theta\gamma_5} \tag{17c}$$

in the mass term $m\overline{\psi}\psi$ would constitute a flaw in the model. However, spontaneous symmetry breaking can be used to circumvent this difficulty. The interaction

$$\mathcal{L}_I = g\overline{\psi}\psi\, A + i\overline{\psi}\gamma_5\psi B = g\overline{\psi}[\phi P_+ + \phi_+ P_-]\psi \tag{18}$$

[where $\phi = A + iB$ is a complex scalar and $P_\pm = \frac{1}{2}(1 \pm \gamma_5)$ are a complete set of orthonormal projection operators] is invariant under the gauge transformation of (17) provided

$$\phi \to e^{-2i\theta}\phi \tag{19}$$

since $e^{i\alpha\gamma_5} P_\pm = e^{\pm i\alpha} P_\pm$. If now the potential for ϕ is such that it acquires a real vacuum expectation value $\langle A\rangle \neq 0$ so that the field ψ develops a mass $m = g\langle A\rangle$, then there is no explicit breaking of the chiral gauge symmetry. In principle, the vector field A_μ would also become massive due to the Higgs mechanism if coupled to ϕ, but this mass is neglected. Furthermore, we do not consider the effect of any degrees of freedom associated with the scalar field ϕ in (16) on the energy levels of ψ. This is feasible provided the mass of ϕ is sufficiently large.

An exact analysis of scattering is not possible for the potential we have been considering as we do not have an exact solution to the wave equation.

(In quantum electrodynamics, such an analysis is possible: e.g., Chapter 9.9 of ref. 8.) However, one can easily compute the lowest order scattering matrix for unpolarized particles and obtain the analogue of the Rutherford formula,

$$\frac{d\bar{\sigma}}{dr} = \frac{4(16\pi^2\alpha^2)^2 m^2}{2|\bar{q}|^4}\mathrm{Tr}\left[\frac{\not{p}_i + m}{2m}\gamma_0\gamma_5\frac{\not{p}_f + m}{2m}\gamma_0\gamma_5\right]$$
$$= \frac{(16\pi^2\alpha)^2}{4|\bar{p}|^4\sin^4(\theta/2)}\left[\left(1 - \beta^2\sin^2\frac{\theta}{2}\right)E^2 + m^2\right] \tag{20}$$

where the notation of ref. 7 has been used.

One could proceed with a Foldy–Wouthuysen type of transformation to expand the Hamiltonian

$$H = \bar{\alpha}\cdot(\bar{p} - e\bar{A}\gamma_5) + \beta m + e\Phi\gamma_5 \tag{21}$$

in inverse powers of m. However, since gauge invariance is absent when $m \neq 0$, the coefficients of this expansion will not be gauge invariant.

It is also worth noting that the operator relation

$$(\not{p} - \not{A}\gamma_5)^2 = (p_\lambda + i\sigma_{\lambda\mu}A^\mu\gamma_5)^2 - iA^\mu_{,\mu}\gamma_5 + 2A^2 \tag{22}$$

indicates that gauge invariance will be absent in one-loop Green's functions. This can be seen most clearly when one employs the calculational techniques of ref. 10. Further discussion of the significance of Eq. (22) appears in ref. 9.

ACKNOWLEDGMENTS

We would like to thank NSERC for financial support and R. and D. MacKenzie for useful discussions.

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